Noncommutative Schur functions

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What is this?

This is (going to be) an "infinite napkin" set of notes I am taking about the Fomin-Greene theory of noncommutative Schur functions.

Contents

1 Ideals and words

Let $\mathbf{u} = (u_1, \ldots, u_N)$ be a collection of variables. Let $\langle \mathbf{u} \rangle$ be the free semigroup on the generators $\mathbf{u}$. Then, let $\mathcal{U} = \mathbb{Z}\langle \mathbf{u} \rangle$ denote the corresponding semigroup ring— the free associative ring generated by $\mathbf{u}$.

We will denote by $\mathcal{U}^*$ the $\mathbb{Z}$-module spanned by words in the alphabet $\{1, \ldots, N\}$.

We will have a fundamental pairing $\langle -, - \rangle$ given by making noncommutative monomials dual to words.

Now, if I is an ideal of $\mathcal{U}$, we define $I^\perp$ by

$$I^\perp := \{\gamma \in \mathcal{U}^* \mid \langle I, \gamma \rangle = 0\}.$$

2 Noncommutative e's and h's

Definition 2.1. The **noncommutative elementary symmetric function** $e_k(\mathbf{u})$ is defined to be

$$e_k(\mathbf{u}) := \sum_{i_1 > i_2 > \cdots > i_k} u_{i_1} u_{i_2} \cdots u_{i_k}.$$

The **noncommutative complete homogeneous symmetric function** $h_k(\mathbf{u})$ is defined to be

$$h_k(\mathbf{u}) := \sum_{i_1 \geq i_2 \geq \cdots \geq i_k} u_{i_1} u_{i_2} \cdots u_{i_k}.$$

2.1 The ideal I_C

Lemma 2.2. Let I be an ideal of $\mathcal{U}$. The following are equivalent:

(a) $e_k(\mathbf{u}) e_j(\mathbf{u}) \equiv e_j(\mathbf{u}) e_k(\mathbf{u}) \mod I$ for all j, k.

(b) $h_k(\mathbf{u}) h_j(\mathbf{u}) \equiv h_j(\mathbf{u}) h_k(\mathbf{u}) \mod I$ for all j, k.

Definition 2.3. We define the ideal I_C to be the ideal consisting of exactly the elements

$$u_b^2 u_a + u_a u_b u_a - u_b u_a u_b - u_b u_a^2 \qquad (a < b), \qquad (1)$$

$$u_b u_c u_a + u_a u_c u_b - u_b u_a u_c - u_c u_a u_b \qquad (a < b < c), \qquad (2)$$

$$u_c u_b u_c u_a + u_b u_c u_a u_c - u_c u_b u_a u_c - u_b u_c^2 u_a \qquad (a < b < c). \tag{3}$$

Compactly, these are the relations

$$[u_a u_b]u_a \equiv u_b[u_a u_b], \quad [u_a u_c]u_b \equiv u_b[u_a u_c], \quad [u_c u_b]u_c u_a \equiv [u_c u_b]u_a u_c$$

for all $a < b < c$.

Theorem 2.4. I_C is the smallest ideal in which the elementary symmetric functions $e_k(\mathbf{u}_S)$ and $e_\ell(\mathbf{u}_S)$ commute for any k, ℓ, S.

2.2 The map Ψ_I

Theorem 2.5 (Fundamental theorem of symmetric functions)**.** Let $\Lambda(\mathbf{x})$ denote the ring of symmetric polynomials in the commuting variables $\mathbf{x} = (x_1, \ldots, x_n)$. Then

$$\Lambda(\mathbf{x}) \simeq \mathbb{Q}[e_1(\mathbf{x}), e_2(\mathbf{x}), \ldots, e_n(\mathbf{x})].$$

Proof. See Theorem 7.4.4 in [EC2]. One checks that products of the form. One can prove this via the *Gale-Ryser* theorem. □

Corollary 2.6. If I contains I_C, then the map

$$\begin{aligned} \Psi_I : \Lambda_n(\mathbf{x}) &\to \mathcal{U}/I \\ e_k(\mathbf{x}) &\mapsto e_k(\mathbf{u}) \end{aligned}$$

extends to a ring homomorphism.

Proof. Combine Theorems 2.5 and 2.4. □

3 Noncommutative Schur functions

Definition 3.1. Let $I \supseteq I_C$. The **noncommutative Schur function** $\mathfrak{J}(\mathbf{u}) \in \mathcal{U}/I$ is defined to be

$$\mathfrak{J}_\lambda(\mathbf{u}) = \sum_{\pi \in S_t} \mathrm{sgn}(\pi) e_{\lambda_1^\top + \pi(1) - 1}(\mathbf{u}) e_{\lambda_2^\top + \pi(2) - 2}(\mathbf{u}) \cdots e_{\lambda_t^\top + \pi(t) - t}(\mathbf{u}),$$

where $t = \lambda_1$ is the number of parts of $\lambda^\top$. Alternatively, since the h's commute whenever the e's do,

$$\mathfrak{J}_\lambda(\mathbf{u}) = \sum_{\pi \in S_t} \operatorname{sgn}(\pi) h_{\lambda_1+\pi(1)-1}(\mathbf{u}) h_{\lambda_2+\pi(2)-2}(\mathbf{u}) \cdots h_{\lambda_t+\pi(t)-t}(\mathbf{u}).$$

The first definition is based on the **Kostka-Naegelsbach identity**

$$s_\lambda(\mathbf{x}) = \det\left(e_{\lambda_i^\top+j-i}(\mathbf{x})\right)_{i,j=1}^n,$$

and the second is based on the **Jacobi-Trudi identity**

$$s_\lambda(\mathbf{x}) = \det\left(h_{\lambda_i+j-i}(\mathbf{x})\right)_{i,j=1}^n.$$

Since these are purely polynomials of elementary symmetric and complete homogeneous polynomials, one sees the following

Definition 3.2. If $I \supseteq I_C$, then

$$\Psi_I(s_\lambda(\mathbf{x})) \equiv \mathfrak{J}_\lambda(\mathbf{u}) \mod I.$$

Proof.

$$\begin{aligned}
\Psi_I(s_\lambda(\mathbf{x})) &= \Psi_I\left(\det\left(e_{\lambda_i^\top+j-i}(\mathbf{x})\right)_{i,j=1}^n\right) \\
&= \Psi_I\left(\sum_{\pi \in S_n} \operatorname{sgn}(\pi) h_{\pi_1+\pi(1)-1}(\mathbf{x}) \cdots h_{\pi_n+\pi(n)-n}(\mathbf{x})\right) \\
&\equiv \sum_{\pi \in S_n} \operatorname{sgn}(\pi) h_{\pi_1+\pi(1)-1}(\mathbf{u}) \cdots h_{\pi_n+\pi(n)-n}(\mathbf{u}) \mod I \\
&\equiv \mathfrak{J}_\lambda(\mathbf{u}) \mod I.
\end{aligned}$$

□

Theorem 3.3. If I contains I_C, then for all $\gamma \in I_C^\perp$,

$$\left\langle \prod_{i=1}^m \prod_{j=n}^1 (1 + x_i u_j), \gamma \right\rangle = \sum_\lambda s_\lambda(\mathbf{x}) \left\langle \mathfrak{J}_{\lambda^\top}(\mathbf{u}), \gamma \right\rangle.$$

Proof. □

Theorem 3.4 ([FG98], [BF16]). In the ideal $I_\varnothing$,

$$\mathfrak{J}_\lambda(\mathbf{u}) := \sum_{T \in \mathrm{SSYT}(\lambda;N)} \mathbf{u}^{\mathsf{colword}\, T}.$$

3.1 Cauchy kernel

Definition 3.5. Let $\mathbf{x} = (x_1, x_2 \ldots)$ be a countable collection of commuting variables.

4 Applications

5 Linear programming

Consider the positive cones $\mathcal{U}_{\geq 0}$ and $\mathcal{U}^*_{\geq 0}$.

6 Algebras of operators

Definition 6.1. A **combinatorial representation** of $\mathcal{U}/I$ is

Definition 6.2.

7 Appendix

7.1 Gessel's fundamental quasisymmetric function

Definition 7.1. Let w be a word. We define the **fundamental quasisymmetric function** $Q_{\mathrm{Des}(\mathsf{w})}$ by

$$Q_{\mathrm{Des}(\mathsf{w})} := \sum_{\substack{1 \leq i_1 \leq \cdots \leq i_n \\ j \in \mathrm{Des}(\mathsf{w}) \implies \mathsf{i_j < i_{j+1}}}} x_{i_1} \cdots x_{i_n}.$$

7.2 The Edelman-Greene correspondence

References

[EC2] Richard P. Stanley, *Enumerative Combinatorics. Volume 2*, Cambridge University Press 2023.

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