exp is the inverse of log

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1 Introduction

Definition 1.1. The power series exp and log are defined

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}, \tag{1}$$

$$\log(x) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}. \tag{2}$$

We wish to show the following:

Theorem 1.2.

$$(\exp \circ \log)(x) = (\log \circ \exp)(x) = x. \tag{3}$$

You can show this using Lagrange inversion. However, we will do this with our bare hands.

Here is a small amount of some "boilerplate" we'll need:

Definition 1.3. Let $a_1, \ldots, a_k \geq 0$, and let $n = a_1 + \cdots + a_k$. The corresponding *multinomial coefficient* for this tuple is

$$\binom{n}{a_1, \ldots, a_k} := \frac{n!}{a_1! \cdots a_k!}.$$

Proposition 1.4. The multinomial coefficient

$$\binom{n}{a_1, \ldots, a_k} := \frac{n!}{a_1! \cdots a_k!}$$

counts the number of partitions of the set $\{1, \ldots, k\}$ into parts $A_1, \ldots, A_k$, such that $|A_1| = a_1, \ldots, |A_k| = a_k$.

Proposition 1.5 ("Power rule").

$$\left(\sum_{m=1}^{\infty} c_m x^m\right)^n = \sum_{m=0}^{\infty} \left[\sum_{\substack{a_1, \ldots, a_n \geq 1 \\ a_1 + \cdots + a_n = m}} c_{a_1} \cdots c_{a_n} \right] x^m \tag{4}$$

2 Proof

We first grind out a raw expression for the coefficients of $\log \circ \exp$, *as an exponential generating function*, then demonstrate that each coefficient is a signed sum of *cyclically ordered set partitions*— we then demonstrate cancellation.

Proof of Theorem 1.2.

$$\underbrace{\log}_{\text{expand}} \Big(\exp(x) \Big)$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\Big(\overbrace{\exp(x)}^{\text{expand}} - 1 \Big)^n}{n}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\overbrace{\left(\sum_{m=1}^{\infty}\frac{x^m}{m!}\right)^n}^{\text{Use “power rule” (4)}}}{n}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\left[\sum_{m=0}^{\infty}\sum_{\substack{a_1,\dots,a_n\geq 1\\ a_1+\cdots+a_n=m}}\frac{1}{a_1!\cdots a_n!}x^m\right]}{\underbrace{n}_{\text{break up fraction}}}$$

$$= \sum_{n=1}^{\infty}\underbrace{\frac{(-1)^{n+1}}{n}}_{\text{move into innermost sum}}\sum_{m=0}^{\infty}\sum_{\substack{a_1,\dots,a_n\geq 1\\ a_1+\cdots+a_n=m}}\frac{1}{a_1!\cdots a_n!}x^m$$

$$= \underbrace{\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}}_{\text{interchange sums}}\sum_{\substack{a_1,\dots,a_n\geq 1\\ a_1+\cdots+a_n=m}}\frac{(-1)^{n+1}}{n}\frac{1}{a_1!\cdots a_n!}x^m$$

$$= \sum_{m=0}^{\infty}\underbrace{\sum_{n=1}^{\infty}\sum_{\substack{a_1,\dots,a_n\geq 1\\ a_1+\cdots+a_n=m}}}_{\text{this quantifies over all \textit{strong compositions of } } m}\frac{(-1)^{n+1}}{n}\frac{1}{a_1!\cdots a_n!}x^m$$

$$= \sum_{m=0}^{\infty}\sum_{\substack{a_1,\dots,a_n \text{ is a}\\ \text{strong composition of } m}}\frac{(-1)^{n+1}}{n}\underbrace{\frac{1}{a_1!\cdots a_n!}x^m}_{\text{multiply by } 1=m!/m!}$$

$$= \sum_{m=0}^{\infty}\sum_{\substack{a_1,\dots,a_n \text{ is a}\\ \text{strong composition of } m}}\frac{(-1)^{n+1}}{n}\underbrace{\frac{m!}{a_1!\cdots a_n!}}_{=\binom{m}{a_1,\dots,a_n}}\frac{x^m}{m!}$$

$$= \sum_{m=0}^{\infty}\sum_{\substack{a_1,\dots,a_n \text{ is a}\\ \text{strong composition of } m}}\frac{(-1)^{n+1}}{n}\underbrace{\binom{m}{a_1,\dots,a_n}}_{\text{convert multiplicity into summation}}\frac{x^m}{m!}$$

$$= \sum_{m=0}^{\infty} \underbrace{\sum_{\substack{a_1,\ldots,a_n \text{ is a} \\ \text{strong composition of } m}} \sum_{\substack{A_1,\ldots,A_n \text{ is a partition} \\ \text{of } \{1,\ldots,m\} \\ |A_1|=a_1,\ldots,|A_n|=a_n}}}_{\text{this quantifies over all \textit{ordered set partitions} of} \{1,\ldots,m\}} \frac{(-1)^{n+1}}{n} \frac{x^m}{m!}$$

$$= \sum_{m=0}^{\infty} \sum_{\substack{A_1,\ldots,A_n \text{ is a partition} \\ \text{of } \{1,\ldots,m\}}} \underbrace{\frac{(-1)^{n+1}}{n}}_{\substack{\text{this adds a cyclic symmetry,} \\ \text{which we will now prove}}} \frac{x^m}{m!}$$

$$= \sum_{m=0}^{\infty} \underbrace{\sum_{\substack{A_1,\ldots,A_n \text{ is a partition} \\ \text{of } \{1,\ldots,m\}}}}_{\text{break apart parts and order}} \frac{(n-1)!}{n!} (-1)^{n+1} \frac{x^m}{m!}$$

$$= \sum_{m=0}^{\infty} \sum_{\substack{\mathbf{A} \text{ is an unordered} \\ \text{partition of } \{1,\ldots,m\}}} \underbrace{\sum_{\substack{A_1,\ldots,A_n \text{ is an ordering} \\ \text{of } \mathbf{A}\text{'s parts}}}}_{\text{there are } n! \text{ such orders}} \frac{(n-1)!}{n!} (-1)^{n+1} \frac{x^m}{m!}$$

$$= \sum_{m=0}^{\infty} \sum_{\substack{\mathbf{A} \text{ is an unordered} \\ \text{partition of } \{1,\ldots,m\}}} n! \frac{(n-1)!}{n!} (-1)^{n+1} \frac{x^m}{m!}$$

$$= \sum_{m=0}^{\infty} \sum_{\substack{\mathbf{A} \text{ is an unordered} \\ \text{partition of } \{1,\ldots,m\}}} \underbrace{(n-1)!}_{\substack{\text{convert multiplicity into sum— this is} \\ \text{the \# of cyclic orders of length } n}} (-1)^{n+1} \frac{x^m}{m!}$$

$$= \sum_{m=0}^{\infty} \sum_{\substack{\mathbf{A} \text{ is an unordered} \\ \text{partition of } \{1,\ldots,m\}}} \sum_{\substack{A_{\underline{1}},\ldots,A_{\underline{n}} \text{ is a cyclic} \\ \text{ordering of } \mathbf{A}\text{'s parts}}} (-1)^{n+1} \frac{x^m}{m!}$$

$$= \sum_{m=0}^{\infty} \underbrace{\left[\sum_{\substack{A_{\underline{1}},\ldots,A_{\underline{n}} \text{ is a cyclically ordered} \\ \text{partition of } \{1,\ldots,m\}}} (-1)^{n+1} \right]}_{\text{voilà!}} \frac{x^m}{m!}$$

Now for this series to be equal to x, it must be that

$$\left[\sum_{\substack{A_{\underline{1}},\ldots,A_{\underline{n}} \text{ is a cyclically ordered} \\ \text{partition of } \{1,\ldots,m\}}} (-1)^{n+1}\right] = \begin{cases} 1, & \text{if } m = 1; \\ 0, & \text{otherwise.} \end{cases}$$

We will prove this using a **sign-reversing involution**— a function which records exactly how terms cancel out in a sum.

> **Cancellation principle.** Let X be a finite set and let $\operatorname{sign} : X \to \mathbb{R}$. If $f : X \to X$ is a bijection such that $\operatorname{sign} f(x) = -\operatorname{sign} f(x)$ for all $x \in X$, then $\sum_{x \in X} \operatorname{sign} x = 0$.

Now, if we define $\operatorname{sign} \mathbf{A} = (-1)^{\operatorname{len} \mathbf{A}+1}$, we can prove that

$$\left[\sum_{\substack{\mathbf{A} \text{ is a cyclically ordered} \\ \text{partition of } \{1,\ldots,m\}}} \operatorname{sign}(\mathbf{A})\right] = 0$$

for all $m > 1$ if we can find such a function.

Let f be the function defined by

$$f(\mathbf{A}) = \begin{cases} A_{\underline{k}} \cup A_{\underline{k+1}}, \ldots, A_{\underline{n+k}} & \text{if } A_{\underline{k}} = \{1\} \\ \{1\}, A_{\underline{k}} \setminus \{1\}, \ldots, A_{\underline{n+k}} & \text{if } A_{\underline{k}} \neq \{1\} \end{cases}$$

where $A_{\underline{k}}$ is the part of $\mathbf{A}$ containing 1.

Then, $\operatorname{sign} f(\mathbf{A}) = -\operatorname{sign} f(\mathbf{A})$.

- In the first case, we are merging two parts, $A_{\underline{k}}$ and $A_{\underline{k+1}}$, decreasing the number of parts by one.
- In the second case, we are splitting apart $A_{\underline{k}}$ into two parts— one that contains $\{1\}$ and one that doesn't.

Moreover f is an involution, therefore it is a bijection.

- If $A_{\underline{k}} = \{1\}$, then on applying f it will be merged with the next nonempty part. Upon applying f again, it will be split apart from this part, reversing what happened.

- If $A_{\underline{k}} \neq \{1\}$, then on applying f we get rid of all elements that aren't 1 and put them in a new part. Upon applying f again, these elements will be merged into our original part again, reversing what happened.

Moreover, this map is actually only defined whenever $m > 1$, as in the $m = 1$ case, it's impossible to do any splitting or merging of parts.

This proves that

$$\left[\sum_{\substack{\mathbf{A}\text{ is a cyclically ordered}\\ \text{partition of }\{1,\ldots,m\}}} \operatorname{sign}(\mathbf{A})\right] = \begin{cases} 1, & \text{if } m = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$(\log \circ \exp)(x) = \sum_{m=0}^{\infty} \left[\sum_{\substack{A_{\underline{1}},\ldots,A_{\underline{n}}\text{ is a cyclically ordered}\\ \text{partition of }\{1,\ldots,m\}}} (-1)^{n+1}\right] \frac{x^m}{m!} = x,$$

which completes the proof. □